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ON CONJUGATED PROBLEMS OF HEAT TRANSFER

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Abstract—The paper deals with the simplest "conjugated" boundary value problems of heat transfer in which heat conduction equations are solved in common for a body with heat sources and for a liquid flowing round the body. The method of the asymptotic solution of integral equations occurring in conjugated problems is presented.

Résumé—L'article traite des problèmes aux limites mixtes les plus simples pour le transport de chaleur, dans lesquels les équations de conduction sont résolues simultanément pour un corps comportant des sources de chaleur, et pour un liquide s'écoulant autour du corps.
On utilise la solution asymptotique des équations intégrales mises en jeu dans ces problèmes conjugués,

Zusammenfassung—Die Arbeit behandelt die einfachsten "konjugierten" Grenzwertprobleme des Wärmeübergangs, bei welchen die Wärmeleitungsgleichungen für den Körper mit Wärmequellen und die ihn unströmende Flüssigkeit gemeinsam gelöst werden können. Für die Integralgleichungen, wie sie bei konjugierten Problemen auftreten, wird die Methode der asymptotischen Lösung angegeben.

Аннотация—В работе рассматриваются простейшие «сопряженные» краевые задачи теплообмена, в которых совместно решаются уравнения теплопроводности для тела с источниками тепла и для жидкости, обтекающей тело. Даётся метод асимптотического решения интегральных уравнений, возникающих в сопряженных задачах.

NOMENCLATURE

- $\theta(x,y)$, liquid temperature;
- $t(x,y)$, solid temperature;
- $\Theta(x)$, surface temperature;
- $p(x)$, normal temperature derivative on the body surface;
- u, v, longitudinal (along the solid) and normal velocity components, respectively;
- K_f, thermal conductivity of liquid;
- K_s, thermal conductivity of solid;
- α, local heat transfer coefficient;
- Nu_x, local Nusselt number;
- Re_x, local Reynolds number;
- Pr, Prandtl number.

1. INTRODUCTION

THE local coefficient of heat transfer between a liquid and a body submerged in it is determined as [1],

$$\alpha = \frac{q}{t_w - t_\infty},$$

where

t_w = wall temperature at the considered point,
t_∞ = temperature of the incoming liquid,
q = density of heat flow through the surface of the immersed body.

Consequently, to determine α, temperature distribution in the liquid and the temperature on the surface of the immersed body must be known.

As a rule, surface temperature of the immersed body—or heat flow through the surface—is considered a given function of co-ordinates [2] and, in particular, of t_w = const. [3].

In some cases, the setting of the wall temperature instead of its determination from the common solution of heat conduction equations for a liquid and an immersed body is unsatisfactory because of the following:

(1) the wall temperature should not be assigned in the case of intense heat transfer,

(2) in the usual statement of the problem the solution does not contain the dependence on the properties of the immersed body—its thermal constants, size, etc.,

(3) an experimental definition of the wall temperature as a function of co-ordinates is a comparatively complicated problem, whereas it is easier to determine the distribution of sources in the body.*

Therefore the problems, which we shall call "conjugated", are formulated as follows: the common solution of heat conduction equations for the body and the liquid round it is to be found. The velocity distribution in the moving liquid is found by solving the corresponding hydrodynamic problem.

The present paper deals with the simplest conjugated problems. An exact solution of the problem on heat transfer in a slip flow is given in section 2. In section 3 the problem on heat transfer between a thin plate and a laminar boundary layer of incompressible liquid formed on it is solved. The method of the asymptotic solution of one class of singular integral equations to which the problems of the considered type are reduced is given in the Appendix. Therefore, the same method can be applied to the solution of other conjugated problems.

2. CONJUGATED PROBLEM OF HEAT TRANSFER IN SLIP FLOW

Consider a liquid flowing with a constant velocity around a solid occupying the quandrant $x > 0$, $y < 0$ (see Fig. 1). Assume that the component along the axis y is equal to zero. Moreover, heat conduction of the fluid along the axis x is neglected

$$\left(\frac{\partial^2\theta}{\partial x^2} \ll \frac{\partial^2\theta}{\partial y^2}, \text{ "boundary layer"}\right).$$

Thus, the problem is formulated as follows: [$\theta(x,y)$ is the fluid temperature, $t(x,y)$ is the solid temperature].

* The method of determination of the heat transfer coefficient proposed by Academician Luikov is based on it.

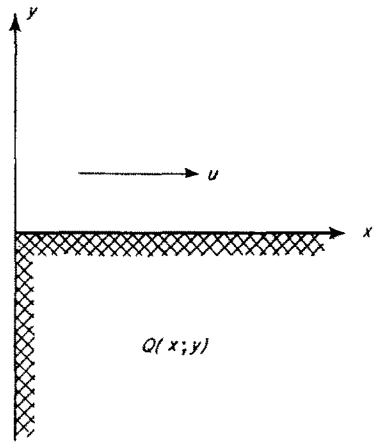

FIG. 1.

For the liquid

$$u\frac{\partial\theta}{\partial x} = \chi\frac{\partial^2\theta}{\partial y^2},$$
$$0 \leqslant x < \infty, \quad 0 \leqslant y < \infty. \tag{1}$$

For the solid

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = -\frac{1}{K_s}Q(x,y),$$
$$0 \leqslant x < \infty, \quad -\infty < y \leqslant 0. \tag{2}$$

Restrictions on the function $Q(x,y)$ to be made for existence of the solution will be found out below.

On the solid–fluid surface we take:†

$$\theta|_{y=+0} = t|_{y=-0}, \tag{3}$$

$$-K_f\frac{\partial\theta}{\partial y}\bigg|_{y=+0} = -K_s\frac{\partial t}{\partial y}\bigg|_{y=-0}. \tag{4}$$

Counting all the temperatures from the temperature of the incoming fluid one may take:

$$\theta|_{x=0} = 0, \tag{5}$$

$$\theta|_{y=\infty} = 0. \tag{6}$$

† Instead of condition (3) we may take a more general condition (temperature jump) $\theta|_{y=+0} - t|_{y=-0} = f(x)$ where $f(x)$ is the arbitrary given function. The solution of the problem with such a condition is quite analogous. For existence of the solution it is necessary that the transformation (14a) for $f(x)$ should exist and $f_s(\alpha)$ should satisfy the same conditions as the function $g(\xi)$ of (18).

Let

$$t|_{x=0} = 0. \tag{7}$$

The conjugated problem formulated is of interest when considering some cases of rarefied gas flow with slip near the wall* [4].

The system of equations (1–7) can also be treated as a boundary value problem for the equation of a mixed type with singular coefficients: an elliptic equation at $y < 0$ and a parabolic one at $y > 0$. The general theory of equations of a mixed type and especially the case of a hyperbolic-elliptic equation is considered in [5].

For equation (1) with conditions (5, 6) and with regard to (4) it is easy to get

$$\theta(x,y) = a_1 \int_0^x \frac{\exp\left[-\dfrac{y^2}{4\gamma(x-x')}\right]}{\sqrt{(x-x')}} p(x')\,\mathrm{d}x', \tag{8}$$

$$\left.\begin{aligned} \left.\frac{\partial t}{\partial y}\right|_{y=-0} &= \frac{1}{\kappa}\left.\frac{\partial \theta}{\partial y}\right|_{y=+0} \equiv -p(x), \\ \kappa = K_s/K_f, \quad a_1 &= \kappa\sqrt{\frac{\gamma}{\pi}}, \quad \gamma = \frac{\chi}{u}. \end{aligned}\right\} \tag{9}$$

For equation (2) with conditions (7) and (9) we have

$$t(x,y) = K(x,y) - \frac{1}{2\pi}\int_0^\infty \ln\frac{(x+x')^2+y^2}{(x-x')^2+y^2} p(x')\,\mathrm{d}x', \tag{10}$$

where

$$K(x,y) = \frac{2}{\pi}\int_0^\infty K_s(\alpha,y)\sin\alpha x\,\mathrm{d}\alpha, \tag{11a}$$

$$K_s(\alpha,y) = \frac{2}{\pi}\int_0^\infty \frac{q_{sc}(\alpha,\beta)}{\alpha^2+\beta^2}\cos\beta y\,\mathrm{d}\beta, \tag{11b}$$

$$q_{sc}(\alpha,\beta) = \textstyle\int_0^\infty q_s(\alpha,-y)\cos\beta y\,\mathrm{d}y, \tag{11c}$$

$$q_s(\alpha,y) = \frac{1}{K_s}\int_0^\infty Q(x,y)\sin\alpha x\,\mathrm{d}x. \tag{11d}$$

It is seen from (8) and (10) that if the normal derivative on the boundary $p(x)$ is known, then the problem is reduced to quadratures. Taking $y = 0$ in equations (8) and (10), the system of integral equations will be obtained

$$\Theta(x) = a_1\int_0^x \frac{p(y)}{\sqrt{(x-y)}}\,\mathrm{d}y, \tag{12}$$

$$\Theta(x) = K(x,0) - \frac{1}{\pi}\int_0^\infty \ln\left|\frac{x+y}{x-y}\right| p(y)\,\mathrm{d}y, \tag{13}$$

where (see equation (3)) the following designation is introduced

$$\theta|_{y=+0} = t|_{y=-0} \equiv \Theta(x). \tag{14}$$

Subject the system (12, 13) to the generalized Fourier sine-transform [6]:

$$f_s(\alpha) = \lim_{\sigma\to 0}\textstyle\int_0^\infty f(x)\,\mathrm{e}^{-\sigma x}\sin\alpha x\,\mathrm{d}x. \tag{14a}$$

It is possible to show the validity of changing the order of integrations shown. Finally, eliminating $\Theta(x)$, a singular equation for $p_s(\alpha)$ is obtained:†

$$\left[1+\frac{1}{a\sqrt{\alpha}}\right]p_s(\alpha) = \frac{\sqrt{\alpha}K_s(\alpha)}{a} - \frac{2}{\pi}⨍_0^\infty \frac{p_s(\beta)\beta}{\beta^2-\alpha^2}\,\mathrm{d}\beta, \tag{15}$$

here $K_s(\alpha) \equiv K_s(\alpha, 0)$ (see (11)); $a = a_1\sqrt{\pi}/2$, ⨍ designates the Cauchy principal value of integral. In equation (15) the change of variables is allowable

$$\alpha = \frac{\sqrt{\xi}}{a^2}, \quad p_s(\alpha) = \varphi(\xi). \tag{16}$$

Equation (15) is reduced to the form

$$\left(1+\frac{1}{\xi^{1/4}}\right)\varphi(\xi) = g(\xi) + \frac{1}{\pi}⨍_0^\infty \frac{\varphi(\eta)}{\xi-\eta}\,\mathrm{d}\eta, \tag{17}$$

where

$$g(\xi) = \frac{\xi^{1/4}K_s(\sqrt{\xi}/a^2)}{a^2}. \tag{18}$$

The general theory of equations of such a type as (17)—singular integral equations with the Cauchy kernel—is considered in [8, 9].

* As suggested by Professor A. A. Gukhman.

† The kernels analogous to those of the integral equation (15) are considered in the Kramers–Kronig transform theory (see, for example, [7]).

Subjecting equation (17) to the Mellin transform and introducing the designation

$$\bar{\varphi}(s) = \int_0^\infty \varphi(\xi)\xi^{s-1}\,d\xi$$

and

$$\bar{g}(s) = \int_0^\infty g(\xi)\xi^{s-1}\,d\xi,$$

we get

$$(1 + ctg\, \pi s)\bar{\varphi}(s) + \bar{\varphi}(s - \tfrac{1}{4}) = \bar{g}(s), \qquad (19)$$

in the range $0 < \mathrm{Re}(s) < 1$.

The latter difference equation for $\bar{\varphi}(s)$ can be reduced to another difference equation with constant coefficients. Omitting simple computations the result is

$$\bar{\varphi}(s) = \bar{\nu}(s) - \tfrac{1}{4}\bar{\varphi}(s - 1),$$
$$\tfrac{3}{4} < \mathrm{Re}(s) < 1, \qquad (20)$$

where

$$\bar{\nu}(s) = \frac{1}{\sqrt{2}} \cdot \frac{\sin \pi s}{\sin \pi (s + \frac{1}{4})}\bar{g}(s) - \frac{1}{2} \cdot \frac{\sin \pi (s - \frac{1}{4})}{\sin \pi (s + \frac{1}{4})}\bar{g}(s - \tfrac{1}{4}) - \frac{1}{2\sqrt{2}} \times \frac{\cos \pi s}{\sin \pi (s + \frac{1}{4})}\bar{g}(s - \tfrac{1}{2}) + \tfrac{1}{4}\bar{g}(s - \tfrac{3}{4}). \qquad (21)$$

The general solution of equation (20) has the form:

$$\bar{\varphi}(s) = \frac{1}{4^s} \cdot \bar{f}(s) + \int_0^\infty \frac{\nu(\xi)\xi^s}{\xi + \frac{1}{4}}\,d\xi, \qquad (22)$$

where

$$\nu(\xi) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \bar{\nu}(s)\frac{ds}{\xi^s}, \quad |\tfrac{3}{4} < \sigma < 1|,$$

and $\bar{f}(s)$ is the arbitrary function satisfying the condition

$$\bar{f}(s) = -\bar{f}(s + 1).$$

The function $\bar{f}(s)$ in (22) is chosen to satisfy original equation (19).

$\bar{\varphi}(s)$ being determined, the problem is solved and reduced to the performance of the inverse Mellin transform. The unknown function $p(x)$ will be determined:

$$p(x) = \lim_{\sigma \to 0} \frac{2}{\pi}\int_0^\infty e^{-\sigma\alpha}\,\varphi(a^4\alpha^2)\sin \alpha x\,d\alpha.$$

$\Theta(x)$ can be found, for example, from (12).

For illustration, the results will be given when the source has the form:

$$Q(x, y) = Q_0\frac{\cos \omega y}{\sqrt{x}}. \qquad (23)$$

The general solution of equation (19) is obtained by the method given above

$$\bar{\varphi}(s) = \frac{\bar{\omega}(s)}{4^s \,.\, \sin \pi (s + \frac{1}{4})} + \frac{q_0}{1 + \sqrt{(2)}\, b^{1/4}} \cdot \frac{\pi b^{s-3/4}}{\sin \pi (s + \frac{1}{4})}, \qquad (24)$$

where

$$q_0 = \frac{Q_0}{K_s} \cdot \sqrt{\left(\frac{\pi}{2}\right)}\, a^3, \quad b = \omega^2 \,.\, a^4$$

and $\bar{\omega}(s)$ is the arbitrary function satisfying

$$\bar{\omega}(s) = -\bar{\omega}(s - \tfrac{1}{4}).$$

Apparently we are to take $\bar{\omega}(s) \equiv 0$. Then

$$\varphi(\xi) = \xi^{1/4} \cdot \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \bar{\varphi}(s - \tfrac{1}{4})\frac{ds}{\xi^s} = \frac{q_0}{1 + \sqrt{(2)}\, b^{1/4}} \cdot \frac{\xi^{1/4}}{\xi + b}, \quad |\tfrac{3}{4} < \sigma < 1|$$

and for $p(x)$ and $\Theta(x)$ we get, respectively:

$$p(x) = \sqrt{\left(\frac{2}{\pi}\right)} \cdot \frac{Q_0}{K_s} \times \frac{1}{1 + \sqrt{(2\omega)}\, a}\int_0^\infty \frac{\sqrt{\alpha}\sin \alpha x}{\alpha^2 + \omega^2}\,d\alpha, \qquad (25)$$

$$\Theta(x) = \sqrt{\left(\frac{2}{\pi}\right)} \cdot \frac{Q_0}{K_s} \times \frac{\sqrt{(2\omega)}\, a}{1 + \sqrt{(2\omega)}\, a}\int_0^\infty \frac{\sin \alpha x}{\sqrt{\alpha}\,(\alpha^2 + \omega^2)}\,d\alpha. \qquad (26)$$

Note, that the integrals in formulae (25) and (26) converge uniformly relative to x, which makes the study of their properties easier.

3. HEAT TRANSFER IN A LAMINAR BOUNDARY LAYER OF FLOW ROUND A THIN PLATE WITH INTERNAL HEAT SOURCES

Consider heat transfer between a thin plate with internal heat sources and a laminar boundary layer of incompressible liquid formed on it. Let $t(x,y)$ and $\theta(x,y)$ be temperatures of the plate and liquid, respectively, and $u(x,y)$ and $v(x,y)$ the components of the velocity of a liquid along the axes x and y (Fig. 2). The system of equations for the considered problem is:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \quad (27)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (28)$$

with the boundary conditions

$$u|_{y=0} = v|_{y=0}, \quad u|_{y=\infty} = U. \quad (29)$$

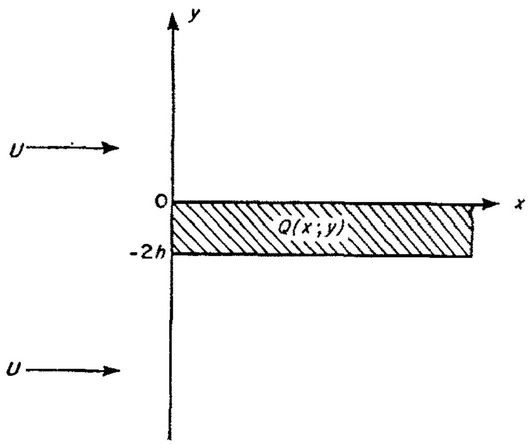

FIG. 2.

For fluid temperature we have

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \chi\frac{\partial^2 \theta}{\partial y^2}. \quad (30)$$

If we take the temperature of the incoming liquid as zero, then

$$\text{(a)} \quad \theta|_{x=0} = 0, \quad \text{(b)} \quad \theta|_{y=\infty} = 0. \quad (31)$$

For the plate

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = -\frac{1}{K_s}\cdot Q(x,y). \quad (32)$$

Restricting ourselves to the sources of the finite power, for which

$$\int Q(x,y)\,\mathrm{d}V < \infty \quad (33)$$

(integration over the whole volume of the plate). All given below is applicable for any source satisfying condition (33). To illustrate, consider the source of the form:

$$Q(x,y) = Q_0 \,.\, \epsilon(l - x), \quad -2h \leqslant y \leqslant 0, \quad (34)$$

where

$$\epsilon(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0. \end{cases}$$

Due to the symmetry of the problem

$$\left.\frac{\partial t}{\partial y}\right|_{y=-h} = 0. \quad (35)$$

On the leading edge of the plate we take

$$t|_{x=0} = 0. \quad (3)$$

The latter condition is unessential and can be replaced by another one.

Finally, write out the plate–liquid boundary conditions:

$$\theta|_{y=+0} = t|_{y=-0}, \quad (37)$$

$$-K_f\left.\frac{\partial \theta}{\partial y}\right|_{y=+0} = -K_s\frac{\partial t}{\partial y}_{y=-0}. \quad (38)$$

The exact Blasius solution is known [10] for equations (27, 28) with boundary conditions (29) describing the laminar boundary layer on a flat plate. The problem is similar [11–13]; the variable $\eta = y/2\sqrt{(U/\nu x)}$ being introduced, it is reduced to the ordinary differential equation

$$\frac{\mathrm{d}^3 f}{\mathrm{d}\eta^3} + f\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2} = 0, \quad (39)$$

with boundary conditions

$$f(0) = f'(0) = 0, \quad f'(\infty) = 2,$$

where $f(\eta)$ is bound with the flow function $\psi(x,y)|u = \partial\psi/\partial y,\ v = -\partial\psi/\partial x|$ by the relation $\psi = \sqrt{(\nu U x)}f$. The solution of equation (39) can be obtained numerically or in the form of power series by η. The latter has the form

$$f(\eta) = \frac{c\eta^2}{2!} - \frac{c^2\eta^5}{5!} + \ldots \quad (40)$$

here $c \equiv f''(0) = 1{\cdot}328$.

Let us turn to equation (30). We introduce a new independent variable ψ (flow function) instead of y following [13]. Since ψ is determined to within the arbitrary constant, we take it that when y is equal to zero then $\psi = 0$ as well. Equation (30) and condition (38) will be written:

$$\frac{\partial\theta}{\partial x} = \chi \frac{\partial}{\partial\psi}\left[u\frac{\partial\theta}{\partial\psi}\right], \tag{41}$$

$$u\left.\frac{\partial\theta}{\partial\psi}\right|_{\psi=0} = -\kappa\, p(x), \tag{42}$$

where

$$p(x) \equiv -\left.\frac{\partial t}{\partial y}\right|_{y=0} \text{ and } \kappa = \frac{K_s}{K_f}.$$

If the Prandtl number is not low, then we may restrict ourselves to the first term in expansion (40). Introducing a new variable [14]

$$z = \tfrac{2}{3}\,\psi^{3/4},$$

we get for (41, 42 and 31), respectively,

$$\frac{4}{\beta}x^{1/4}\frac{\partial\theta}{\partial x} = \frac{\partial^2\theta}{\partial z^2} + \frac{1}{3z}\cdot\frac{\partial\theta}{\partial z}, \tag{43}$$

$$\frac{3^{1/3}\beta}{2^{4/3}\chi}\cdot\frac{z^{1/3}}{x^{1/4}}\cdot\left.\frac{\partial\theta}{\partial z}\right|_{z=0} = -\,\kappa\, p(x), \tag{44}$$

$$\text{(a)}\quad \theta|_{x=0} = 0, \qquad \text{(b)}\quad \theta|_{z=\infty} = 0, \tag{45}$$

where

$$\beta = \frac{\chi\sqrt{(c/2)}U^{3/4}}{\nu^{1/4}}.$$

Equation (43) was studied in [15]; the solution has the form:

$$\left.\begin{aligned}\theta(x,z) = {}& \frac{3z^{1/3}}{2\beta x^{3/4}}\cdot\exp\left(-\frac{3z^2}{4\beta x^{3/4}}\right)\\ & \left\{\int_0^\infty f_1(\lambda)\exp\left(-\frac{3\lambda^2}{4\beta x^{3/4}}\right)\right.\\ & \times I_{1/3}\left(\frac{3\lambda z}{2\beta x^{4/3}}\right)\lambda^{2/3}\,\mathrm{d}\lambda\\ & + \int_0^\infty f_2(\lambda)\exp\left(-\frac{3\lambda^2}{2\beta x^{3/4}}\right)\\ & \left. I_{-1/3}\left(\frac{3\lambda z}{2\beta x^{4/3}}\right)\lambda^{2/3}\,\mathrm{d}\lambda\right\}.\end{aligned}\right\} \tag{46}$$

The functions $f_1(\lambda)$ and $f_2(\lambda)$ are determined from the boundary conditions. Condition (45a) gives $f_1(\lambda) = -f_2(\lambda) \equiv f(\lambda)$ and it follows from (45b) that $f(\infty) = 0$.

Introducing the designation

$$\theta|_{z=+0} \equiv \Theta(x) \tag{47}$$

and making use of condition (44) it is easy to get

$$\Theta(x) = -\frac{a}{x^{1/2}}\int_0^\infty \psi(u)\exp\left[-\frac{1}{\beta}\left(\frac{u}{x}\right)^{3/4}\right]\frac{\mathrm{d}u}{u^{1/2}} \tag{48}$$

$$p(x) = -\frac{b}{x^{5/4}}\int_0^\infty \psi(u)\exp\left[-\frac{1}{\beta}\left(\frac{u}{x}\right)^{3/4}\right]\frac{\mathrm{d}u}{u^{1/4}}, \tag{49}$$

where

$$\psi(u) \equiv f\left(\frac{2}{\sqrt{3}}u^{3/8}\right),\ a = \frac{3}{4\Gamma(\frac{2}{3})\beta^{2/3}},$$

$$b = \frac{3^{5/3}}{8\Gamma(\frac{1}{3})\chi\kappa\beta^{1/3}}.$$

Finally, eliminate $\psi(u)$ from equations (48, 49). Then we have

$$\Theta(x) = \gamma\int_0^x \frac{p(y)\,\mathrm{d}y}{(x^{3/4} - y^{3/4})^{2/3}}, \tag{50}$$

here

$$\gamma = \frac{3a}{4\Gamma(1/3)b\beta^{1/3}}.$$

Consider equation (32). The solution of it expressed through $p(x)$ at conditions (34–36) may easily be obtained by the standard methods. We shall directly write the expression for the temperature of the plate surface [see (37, 38 and 47)]:

$$\begin{aligned}\Theta(x) = {}& \frac{Q_0}{K_s}\,x\left(l - \frac{x}{2}\right)\epsilon\,(l - x)\\ & + \frac{Q_0}{K_s}\,\frac{l^2}{2}\,\epsilon(x - l) - \frac{1}{h}\int_0^\infty G(x,y)p(y)\,\mathrm{d}y\\ & - \frac{1}{\pi}\int_0^\infty \ln\left[\frac{1 - \exp\left(-\pi\dfrac{x+y}{h}\right)}{1 - \exp\left(-\pi\dfrac{|x-y|}{h}\right)}\right]p(y)\,\mathrm{d}y,\end{aligned} \tag{51}$$

where

$$G(x,y) = \begin{cases} y & y < x \\ x & y > x. \end{cases} \quad (52)$$

Thus, the problem is brought to the solution of the system of integral equations (50, 51) for unknown functions $\Theta(x)$ and $p(x)$. The knowledge of any of them reduces the determination of $\theta(x,y)$ and $t(x,y)$ to the quadratures.

The system (50, 51) cannot be solved exactly, and moreover, the exact solution need not be given, since the equations of the boundary layer theory (27–29) are applicable only at a distance from the leading edge of the plate; otherwise the complete Navier–Stokes equations should be used. Consequently, the solution is to be sought only in the region $x/h \gg 1$.

The latter remark allows basic simplification of equation (51). Accurate within the terms

$$\sim \text{const. } \exp\left(-\frac{\pi x}{h}\right)$$

we immediately obtain

$$\Theta(x) = \frac{Q_0}{K_s}\, x\left(l - \frac{x}{2}\right) \epsilon(l - x)$$

$$+ \frac{Q_0}{K_s} \cdot \frac{l^2}{2}\, \epsilon(x - l) - \frac{1}{h}\int_0^\infty G(x,y)p(y)\, \mathrm{d}y$$

$$+ \frac{1}{\pi}\int_0^\infty \ln\left[1 - \exp\left(-\pi\frac{|x-y|}{h}\right)\right] p(y)\, \mathrm{d}y. \quad (53)$$

Further, the last term in the right-hand side of equation (53) can be given in the form:

$$\frac{1}{\pi}\int_0^\infty \ln\left[1 - \exp\left(-\pi\frac{|x-y|}{h}\right)\right] p(y)\, \mathrm{d}y$$

$$= \frac{x}{\pi}\int_0^\infty \ln\left[1 - \exp\left(-\frac{\pi(y+1)}{h}\, x\right)\right]$$

$$\times\, p[x(y+2)]\, \mathrm{d}y + \frac{x}{\pi}\int_1^1 \ln\Big[1$$

$$- \exp\left(-\frac{\pi x}{h}|y|\right)\Big]\, p[x(y+1)]\, \mathrm{d}y.$$

The first of these two integrals can be easily evaluated at high x/h by the Laplace method. It turns out to be equal to

$$-\frac{h}{\pi^2}\exp\left(-\frac{\pi x}{h}\right) p(2x)$$

and should be neglected. For the evaluation of the second integral we show that at

$$\frac{x}{h} \to \infty\; F\left(y; \frac{x}{h}\right) =$$

$$-\frac{3x}{\pi h}\ln\left[1 - \exp\left(-\frac{\pi x}{h}|y|\right)\right]$$

has the limit of $\delta(y)$ in the sense of distribution. It is necessary to show [16] for this, firstly, that $\int_a^b F(y; x/h)\, \mathrm{d}y$ is limited from above by a constant independent of a, b and x/h at any a and b. Really, since the function $F(y;\, x/h)$ is non-negative, then

$$\int_b^a F\left(y; \frac{x}{h}\right) \mathrm{d}y \leqslant \int_{-\infty}^\infty F\left(y; \frac{x}{h}\right) \mathrm{d}y = 1.$$

Secondly, at any a and b different from zero

$$\lim_{(x/h)\to\infty}\int_b^a F\left(y; \frac{x}{h}\right) \mathrm{d}y$$

$$= \begin{cases} 0 & \text{at } a < b < 0 \text{ and } 0 < a < b \\ 1 & \text{at } a < 0 < b. \end{cases}$$

should be fulfilled, which is easy to check.

Using the latter remark and eliminating $\Theta(x)$ from (50) and (53) we obtain the equation for $p(x)$:

$$\gamma\int_0^x \frac{p(y)\, \mathrm{d}y}{(x^{3/4} - y^{3/4})^{2/3}} = \frac{Q_0}{K_s}\, x\left(l - \frac{x}{2}\right) \epsilon(l - x)$$

$$+ \frac{Q_0}{K_s} \cdot \frac{l^2}{2}\, \epsilon\,(x - l) - \frac{1}{h}\int_0^\infty G(x,y)p(y)\, \mathrm{d}y$$

$$-\frac{h}{3}p(x). \quad (54)$$

It is easy to see from equation (54) that $p(x)$ depends on the dimensionless variable

$$\sim \frac{Re_x^{1/2}Pr^{1/3}}{\kappa} \cdot \frac{x}{h}$$

where $Re_x = Ux/\nu$ is the local Reynolds number and $Pr = \nu/\chi$ is the Prandtl number. Since Pr and Re_x are not low, then it is sufficient to find the asymptotic solution of equation (54) due to the condition $x/h \gg 1$. For this purpose, having subjected equation (54) to the Mellin transform we obtain the difference equation

$$(\bar{p}(s) = \int_0^\infty p(x)x^{s-1}\,dx):$$

$$\tfrac{4}{3}\Gamma(\tfrac{1}{3})\gamma \cdot \frac{\Gamma(2-\frac{4}{3}s)}{\Gamma(\frac{7}{3}-\frac{4}{3}s)}\bar{p}(s-\tfrac{1}{2}) =$$

$$-\frac{Q_0}{K_s}\cdot\frac{l^{s+1}}{s(s-1)(s+1)}$$

$$+\frac{1}{h}\cdot\frac{1}{s(s-1)}\bar{p}(s+1) - \frac{h}{3}\bar{p}(s-1), \qquad (55)$$

where

$$0 < \mathrm{Re}(s) < 1.$$

According to the method of the asymptotic solution of some integral equations given in the Appendix a new function $\bar{\varphi}(s)$ will be substituted for $\bar{p}(s)$:

$$\bar{p}(\tfrac{3}{4}s - 1) = \bar{\Omega}(s)\bar{\varphi}(s), \quad |\mathrm{Re}(s) > 0|, \qquad (56)$$

where

$$\bar{\Omega}(s) = \frac{\Gamma(2s)\Gamma(2s+\frac{2}{3})\Gamma(s+\frac{5}{3})\Gamma(s+\frac{1}{3})\Gamma(s+2)}{\Gamma(\frac{8}{3}-s)}. \qquad (57)$$

(note, that $\bar{\Omega}(s)$ is analytical in the half-plane $\mathrm{Re}(s) > 0$). The proof of the fact that the function $\varphi(x)$ may be looked for in the form of expansion of x inverse power will be omitted, and the asymptotic expansion for $p(x)$ and $\Theta(x)$ will be given directly. For the sake of brevity we shall deduce the results only for the physically interesting region $x \leqslant l$ where the heat release is going on; l is supposed to be sufficiently large, so that inequalities are taking place

$$h \text{ and } \frac{\kappa}{Re_x^{1/2}Pr^{1/3}}\cdot h \ll x \leqslant l. \qquad (58)$$

$$p(x) = \tfrac{3}{4}\sum_{n=0}^{\infty}\frac{a_n}{x^{n/2}}\bar{\Omega}(\tfrac{2}{3}n+\tfrac{4}{3}), \qquad (59)$$

$$\Theta(x) = \Gamma(\tfrac{1}{3})\gamma\sum_{n=0}^{\infty}\frac{a_n}{x^{n/2-1/2}}\bar{K}(\tfrac{2}{3}n+\tfrac{4}{3}), \qquad (60)$$

where $\bar{\Omega}(s)$ is determined by the relation (57) and

$$\bar{K}(s) = \frac{\Gamma(2s)\Gamma(2s+\frac{2}{3})\Gamma(s+\frac{5}{3})\Gamma(s+\frac{1}{3})\Gamma(s+2)}{\Gamma(3-s)}. \qquad (61)$$

The coefficients a_n satisfy the recurrence relation

$$\left.\begin{aligned}
&\bar{K}_1(\tfrac{2}{3}n-\tfrac{4}{3})a_{n-3} - \frac{1}{H}\bar{K}_2(\tfrac{2}{3}n-\tfrac{4}{3})a_n - B\bar{K}_3(\tfrac{2}{3}n-\tfrac{4}{3})a_{n-4} = 0, \quad |n = 1, 2, \ldots|;\\
&a_K \equiv 0 (K < 0);\\
&a_0 = \frac{3^2}{2\cdot 5\cdot 7\,\Gamma^2(\frac{5}{3})\Gamma(\frac{10}{3})}\cdot\frac{Q_0}{K_s}\cdot h.
\end{aligned}\right\} \qquad (62)$$

In equation (62) the following designations are used:

$$\left.\begin{aligned}
\bar{K}_1(s) &= \frac{\Gamma(2s+\frac{4}{3})\Gamma(2s+2)\Gamma(s+\frac{7}{3})\Gamma(s+1)\Gamma(s+\frac{8}{3})}{\Gamma(\frac{4}{3}-s)}\cdot s,\\
\bar{K}_2(s) &= \frac{\Gamma(2s+\frac{16}{3})\Gamma(2s+6)\Gamma(s+\frac{13}{3})\Gamma(s+3)\Gamma(s+\frac{14}{3})}{\Gamma(1-s)}\cdot s,\\
\bar{K}_3(s) &= \frac{\Gamma(2s+1)\Gamma(2s+\frac{2}{3})\Gamma(s+\frac{5}{3})\Gamma(s+\frac{1}{3})\Gamma(s+2)}{\Gamma(\frac{8}{3}-s)}(s-\tfrac{4}{3}),\\
H &= \tfrac{3}{4}\Gamma(\tfrac{1}{3})h\gamma, \quad B = h/4\Gamma(\tfrac{1}{3})\gamma.
\end{aligned}\right\} \qquad (63)$$

From (62) and (63) the coefficients a_n are easily evaluated. We shall write out in an explicit form the first terms of the expansions (59) and (60)

$$p(x) = \frac{Q_0}{K_s} h \left\{ 1 - \frac{1}{2^{5/3} \cdot 3^{2/3} c^{1/3}} \left[\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})} \right]^2 \frac{\kappa}{Re_x^{1/2} Pr^{1/3}} \cdot \frac{h}{x} - \frac{1}{2^{1/3} \cdot 3^{4/3} c^{2/3}} \left[\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})} \right]^4 \frac{\kappa^2}{Re_x Pr^{2/3}} \left(\frac{h}{x} \right)^2 + 0\,(1/x^{7/2}) \right\} \quad (64)$$

$$\Theta(x) = \frac{Q_0}{K_s} \cdot h \frac{\kappa}{Re_x^{1/2} Pr^{1/3}} \cdot x \cdot \frac{2^{1/3}}{3^{2/3} c^{1/3}} \left[\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})} \right]^2 \cdot \left\{ 1 - \frac{1}{2^{5/3} \cdot 3^{2/3} c^{1/3}} \left[\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})} \right]^2 \frac{\kappa}{Re_x^{1/2} Pr^{1/3}} \cdot \frac{h}{x} - \frac{7}{2^{4/3} \cdot 3^{4/3} \cdot 5 \cdot c^{2/3}} \left[\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})} \right]^4 \frac{\kappa^2}{Re_x Pr^{2/3}} \left(\frac{h}{x} \right)^2 + 0\,(1/x^3) \right\}. \quad (65)$$

Hence, for the coefficient of heat transfer and for the local Nusselt number it is easy to get:

$$\alpha = \frac{3^{2/3} c^{1/3}}{2^{1/3}} \left[\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} \right]^2 K_f \frac{Re_x^{1/2} \cdot Pr^{1/3}}{x} - \frac{3^{1/3}}{2^{5/3} \cdot 5 \cdot c^{1/3}} \left[\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})} \right]^2 K_s \left(\frac{h}{x} \right)^2 \frac{1}{Re_x^{1/2}\, Pr^{1/3}} \frac{1}{x} + 0(1/x^4), \quad (66)$$

$$\frac{Nu_x}{Re_x^{1/2}} = \frac{3^{2/3} c^{1/3}}{2^{1/3}} \left[\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} \right]^2 Pr^{1/3} - \frac{3^{1/3}}{2^{5/3} \cdot 5 \cdot c^{1/3}} \left[\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})} \right]^2 \frac{\kappa^2}{Re_x\, Pr^{1/3}} \left(\frac{h}{x} \right)^2 + 0\left(\frac{1}{x^{7/2}} \right). \quad (67)$$

In conclusion we shall give once more the range of application of the results obtained:

(a) inequalities (58),
(b) $Pr > 1$,
(c) $Re_x < 3 \cdot 10^5$.

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APPENDIX

A number of problems of mathematical physics (including the conjugated problems considered above, some other problems of the transfer theory and dispersion relations of the quantum field theory) are resolved by the integral equations of the form:

$$a(x)\varphi(x) = g(x) + \int_0^\infty k(x/y) x^\alpha y^\beta \varphi(y)\, \mathrm{d}y, \quad (A1)$$

where $a(x)$ is the finite sum*

* $a(x)$ can also be a linear differential operator of such a type as $\sum_K a_K x^{\gamma_K} (\mathrm{d}^{n_K}/\mathrm{d}x^{n_K})$. Thus, equation (A1) can be integro-differential too.

$$a(x) = \sum_K a_K x^{\gamma_K}. \qquad (A1')$$

Often it is sufficient to get the asymptotic expansion of the solution of this equation.

If as $x \to \infty$ the kernel $k(x)$ does not decrease more quickly than any power of x, then it is impossible to seek $\varphi(x)$ directly in the form of an asymptotic series.

With the help of the method given below the asymptotic solution of equation (A1) may be obtained for many cases at high x.

Subjecting equation (A1) to the Mellin transform we obtain the finite difference equation

$$\sum_K a_K \Phi(s + \gamma_K) = G(s) + K(s + \alpha)\Phi(s + \alpha + \beta + 1), \qquad (A2)$$

taking place in some band of the complex plane s. Without restriction of the generality we may consider that $0 < \mathrm{Re}(s) < \sigma$ where σ is some positive number.

In equation (A2) we designated

$$\Phi(s) = \int_0^\infty \varphi(x) x^{s-1}\, dx$$

and in an analogous way for the other functions.

Introducing a new unknown function $\Psi(s)$ instead of $\Phi(s)$:

$$\Phi(s + \delta) = \Omega(s)\Psi(s) \quad |\mathrm{Re}(s) > 0|, \qquad (A3)$$

where δ is the smallest number out of γ_K and $\alpha + \beta + 1$. Equation (A2) will take the form

$$\sum_K a_K \Omega(s + \gamma_K - \delta)\Psi(s + \gamma_K - \delta) = G(s) + K(s + \alpha)\Omega(s + \alpha + \beta - \delta + 1) \times \Psi(s + \alpha + \beta - \delta + 1) \quad (A4)$$

The function $\Omega(s)$ is chosen so that, firstly, the inverse Mellin transforms of $\Omega(s)$ as well as of functions $\Omega(s + \gamma_k - \delta)$ and

$$K(s + \alpha)\Omega(s + \alpha + \beta - \delta + 1),$$

being the coefficients in equation (A4) could exist. Secondly, the inverse Mellin transforms of all the functions enumerated must diminish exponentially at $x \to \infty$. To fulfill the latter condition it is necessary that $\Omega(s)$, all the $\Omega(s + \gamma_K - \delta)$ and

$$K(s + \alpha)\Omega(s + \alpha + \beta - \delta + 1)$$

have no singularities in the half-plane $\mathrm{Re}(s) > 0$. Whether it is sufficient depends on the right choice of $\Omega(s)$.

Now we may seek the function $\psi(x)$ in the form of asymptotic expansion, for example,

$$\psi(x) = \sum_{n=0}^{\infty} \frac{c_n,}{x^{an+b}}$$

and obtain the recurrence relations for c_n from equation (A4). The unknown function $\varphi(x)$ is easily determined from equation (A3).

For concrete integral equations it is not difficult to select the function $\Omega(s)$ so that it may have the required analytical properties.

As an example consider a simple Volterra equation of the second kind

$$\varphi(x) = 1 - \int_0^x \frac{\varphi(y)}{\sqrt{(x - y)}}\, dy, \qquad (A5)$$

which belongs to the class of equations (A1). Equation (A5) is taken only for the sake of illustration, since by the standard method it is easy to get its exact solution with the help of the Laplace transformation

$$\varphi(x) = \exp(\pi x)\, \mathrm{erfc}\, [\sqrt{(\pi x)}], \qquad (A6)$$

where

$$\mathrm{erfc}\,(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-t^2)\, dt.$$

Briefly, so as not to introduce the generalized (half-plane) Mellin transforms, we shall present a nonuniform term of equation (A5), for example, in the form of $\lim_{\epsilon \to 0} \exp(-\epsilon\sqrt{x})$ and turn to the limit only at the end.

From equation (A5) we get

$$\lim_{\epsilon \to 0} 2\epsilon^{-2s}\Gamma(2s) - \Phi(s) = \sqrt{\pi}\frac{\Gamma(\frac{1}{2} - s)}{\Gamma(1 - s)}\Phi(s + \tfrac{1}{2}), \quad |0 < \mathrm{Re}(s) < \tfrac{1}{2}| \quad (A7)$$

Introduce instead of $\Phi(s)$ a new function

$$\Phi(s) = \frac{\Gamma(2s)\Gamma(s)}{\Gamma(1 - s)}\Psi(s) \quad |\mathrm{Re}(s) > 0|. \qquad (A8)$$

Upon elementary transformations equation (A7) will acquire the form

$$-\lim_{\epsilon \to 0} 2\epsilon^{-2s}\Gamma(-s) - \frac{\Gamma(s)}{s}\Psi(s) =$$

$$2\sqrt{\pi}\Gamma(s+\tfrac{1}{2})\Psi(s+\tfrac{1}{2}), \quad |\mathrm{Re}(s) > 0|. \qquad \text{(A9)}$$

Equations (A8 and A9) are equivalent, respectively

$$\varphi(x) = 2\int_0^\infty J_0(2y^{1/4})K_0(2y^{1/4})\psi\left(\frac{x}{y}\right)\frac{\mathrm{d}y}{y} \qquad \text{(A10)}$$

and

$$2 + \int_0^\infty Ei(-y)\psi\left(\frac{x}{y}\right)\frac{\mathrm{d}y}{y} = 2\sqrt{(\pi x)}\int_0^\infty \exp(-y)\psi\left(\frac{x}{y}\right)\frac{\mathrm{d}y}{y}. \qquad \text{(A11)}$$

Here $J_0(x)$ is the Bessel function of the zero order, $K_0(x)$ is the Macdonald function of the zero order and $Ei(x)$ is the integro-exponential function.

Now, it is evident that $\psi(x)$ can be sought in the form of the asymptotic series

$$\psi(x) = \sum_{n=0}^{\infty} \frac{c_n}{x^{an+b}}. \qquad \text{(A12)}$$

From (A8 and A9) or (A10 and A11) we get $a = b = \frac{1}{2}$ and c_n will satisfy the relation

$$c_n = \frac{1}{\sqrt{\pi}}\cdot\frac{1}{n}\cdot\frac{\Gamma(n/2)}{\Gamma[(n+1)/2]}c_{n-1} \quad |n = 1, 2, \ldots| \qquad \text{(A13)}$$

at

$$c_0 = \frac{1}{\pi}.$$

Finally,

$$\varphi(x) = \sum_{n=0}^{\infty} \frac{c_{2n}}{x^{n+1/2}}\cdot\frac{\Gamma(2n+1)\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2}-n)}. \qquad \text{(A14)}$$

It is easy to check that the series (A14) the coefficients of which are determined from equation (A13) coincide with the known asymptotic expansion of the solution of (A6).